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Pointed bubbles in slow viscous flow

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Inviscid bubbles confined by the slow axisymmetric straining motion of a very viscous fluid are considered for the case when the surface tension is weak. The shape of the bubbles is determined using slender-body theory, and it is found that these bubbles have pointed ends, in agreement with well-established experimental results. The description obtained is invalid within exponentially small neighbourhoods of the ends and a local analysis suggests that the tips are cusp-like. In both the description of the major portion of the bubble and of the ends, there is an apparent non-uniqueness because a certain parameter can take on a countably infinite number of values. This non-uniqueness is not resolved.

1. Introduction

Inviscid bubbles immersed in a viscous fluid, if strained sufficiently, often exhibit pointed ends. Thus Taylor (1934) subjected bubbles to a corner flow and found that, at sufficiently high rates of strain, hitherto smooth bubbles abruptly developed pointed ends. Similar experiments were carried out by Rumscheidt & Mason in 1961 and confirmed this phenomenon. At a less sophisticated level the kitchen experimenter can easily convince himself of the reality of pointed bubbles with the aid of a bottle of syrup.

Despite this unambiguous experimental evidence little attempt has been made to describe such bubbles theoretically. Thus the basic aim of the present paper is to determine whether or not pointed bubbles can be described within the framework of the slow-flow (inertialess) equations. Ideally, we should like to model exactly the experimental situation of both Taylor and Rumscheidt & Mason, comparing the theoretical predictions with their experimental observations. Unfortunately, in these experiments the undisturbed flow far from the bubble was two-dimensional, so that the flow in the neighbourhood of the bubble was genuinely three-dimensional and its mathematical description presents considerable difficulty. In searching for simpler flows, which might model the experiments in a qualitative fashion, we are naturally led to consider either two-dimensional bubbles immersed in a two-dimensional flow or axisymmetric bubbles. In the former case a pointed interface will have a discontinuous tangent along a *line* in three-space whereas in the latter the discontinuity will be confined to a *point*. Since the experimentally generated bubbles have point discontinuities, rather than edge discontinuities, it might be expected that axisymmetric bubbles will model the experiments more faithfully. Richardson (1968) has discussed the possibility of angular interfaces for two-dimensional bubbles and shown that such discontinuities, if they exist, must be genuine cusps (rather than being wedge-shaped). He points out that if the surface tension is not zero there is, associated with an edge discontinuity, a finite force acting on the surrounding fluid of magnitude $2T$ per unit length (T being the surface tension). There is no such force in the case of a point discontinuity, so two-dimensional bubbles are completely unrepresentative of experimentally realizable drops as far as tangent discontinuities are concerned. For this reason we confine our attention to axisymmetric bubbles.

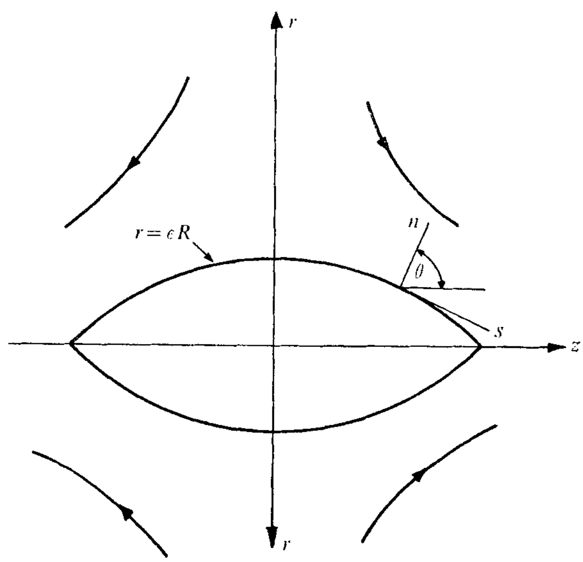

FIGURE 1. Axisymmetric bubble.

Figure 1 shows the physical situation which we wish to describe mathematically. Under the influence of the straining motion the bubble tends to be squeezed out along the z axis (the axis of symmetry). Complete collapse is prevented by surface tension effects, for, the more slender the bubble, the greater is the jump in normal stress across its boundary. Consequently, if the surface tension is small (for an $O(1)$ straining flow), the bubble will be slender and the whole apparatus of slender-body theory can be used. A programme along these lines was proposed by Taylor (1964)† but has apparently never been developed. He gives an approximate analysis, the results of which are correct as far as they go, but the work is not mathematically systematic and some interesting and important features of the analysis are not revealed. Note that if the bubble is pointed it is the interface curvature in planes z = constant that gets larger as the bubble is squeezed, and so provides the resistance to collapse. The curvature in planes of constant azimuth decreases. This supplies another reason for not looking at two-dimensional pointed bubbles, since there is clearly nothing to prevent them being squeezed into flat sheets.

† I am grateful to a referee for bringing this work to my attention.

Slender-body theory generates what is essentially a power series in the surface tension for the bubble shape. The leading term in this series describes pointed bubbles with conical ends, in apparent agreement with experiment. However, subsequent terms contain logarithms, suggesting the presence of non-uniformity within a distance from the ends of $O[\exp(-1/\epsilon^2)]$, where ϵ is the thickness ratio of the bubble. The failure of a local analysis near a conical end provides additional evidence of this non-uniformity. Unfortunately, a description of the bubble in these exponentially small regions has not been obtained, so that we are reduced to speculation about what happens there. In this respect the experimental observations are of little help; for all the experimental bubbles exhibiting pointed ends, ϵ is significantly less than one (roughly a third or smaller) so that these exponentially small regions are not distinguishable.

There are two obvious possibilities for the behaviour very near the ends. It is conceivable that the shape is unsteady, possibly as the result of an instability. Rumscheidt & Mason refer to the shedding of small bubbles from the ends, although it is not clear that this always occurred and, indeed, Taylor's description implies the existence of steady pointed bubbles under some circumstances. Another possibility is that there *is* a steady shape and the exponentially small regions provide a transition from the conical shape of the outer solution to some other shape at the very ends themselves. A necessary condition for this to be true is the existence of a local solution in the immediate vicinity of the end, and it is shown that such a description is possible if the ends are cusp shaped. It certainly cannot be concluded that such cusps are a reality, only that they are a possibility. Another possibility, of course, is that the ends are actually rounded. It should be emphasized that we do not champion cusped ends. The basic concern is with the demonstration that the inertialess equations do admit pointed solutions 'in the large' in agreement with experiment. However, since the non-uniformity calls this analysis into question, it seems necessary to provide at least one plausible resolution of the difficulty. The true resolution must presumably await a careful analysis of the flow near the ends, but how this can be done is not clear.

The bubble shapes obtained, both in the large and locally, near the ends, are not unique. For a given straining flow and surface tension there is a one-parameter family of possible bubble shapes in which the parameter can equal any positive even integer.† No criterion for choosing this parameter has been established, so this problem joins a long list of non-unique free-boundary problems awaiting clarification. The present non-uniqueness is of special interest, however, since the author is not aware of any other steady non-unique slow viscous flows. The present work suggests that it would be futile to try and extend the uniqueness results of Keller, Rubenfeld & Molyneux (1967) to general free-surface flows.

† The striking similarity between the non-uniqueness of the solution in the large and in the local solution provides additional evidence that the cusped shapes are significant.

2. Formulation and leading solution

The Reynolds number is assumed to be so small that the inertia terms can be neglected. The equations of motion are then the axisymmetric Stokes flow equations:

$$\left.\begin{aligned} 0 &= \frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}, \\ 0 &= -\frac{\partial p}{\partial r}+\mu\left(\Delta v_r-\frac{v_r}{r^2}\right), \\ 0 &= -\frac{\partial p}{\partial z}+\mu\Delta v_z, \end{aligned}\right\} \tag{2.1}$$

where $$\Delta \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

These equations admit a solution

$$v_z = Cz, \quad v_r = -\tfrac{1}{2}Cr, \quad p = p_\infty, \tag{2.2}$$

where C and p_∞ are constants, and we choose this as the undisturbed flow far from the bubble. If C is positive this corresponds to the flow sketched in figure 1. The only difference between this and one of the experimental configurations of Taylor is that in Taylor's work the far field is two-dimensional.

The bubble is assumed to be slender and described by

$$r = \epsilon R(z), \tag{2.3}$$

where $\epsilon \ll 1$. The disturbance due to the bubble is then represented by a distribution of irrotational sources and Stokeslets along the z axis between $-a$ and a (the limits of the bubble in the z direction). If the bubble were rounded the singularities would not extend all the way to the ends, but for pointed bubbles, which we anticipate, they do. If we define the stream function in the usual way,

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{2.4}$$

we can write it as

$$\psi = \tfrac{1}{2}Cr^2z+\psi', \tag{2.5}$$

so that ψ' is the disturbance due to the bubble. Since the stream functions generated by a Stokeslet and a source are, respectively,

$$\psi = \frac{r^2}{[r^2+z^2]^{\frac{1}{2}}}, \quad \psi = 1-\frac{z}{[z^2+r^2]^{\frac{1}{2}}}$$

it follows that ψ' has the representation

$$\psi'(z,r) = \int_{-a}^{a} d\tilde{z}\,\frac{f(\tilde{z})\,r^2}{[r^2+(z-\tilde{z})^2]^{\frac{1}{2}}}-\int_{-a}^{a} d\tilde{z}\,\frac{g(\tilde{z})\,(z-\tilde{z})}{[r^2+(z-\tilde{z})^2]^{\frac{1}{2}}}, \tag{2.6}$$

where $$\int_{-a}^{a} d\tilde{z}\,g(\tilde{z}) = 0.$$

This last condition follows from the fact that the bubble is not a net source of fluid. Solution of the problem is equivalent to solution for the source strengths f and g and the bubble shape $R(z)$.

The fluid inside the bubble is assumed to be inviscid and the constant pressure there is arbitrarily assigned the value zero. Mathematically, the bubble then contains a vacuum, so that the boundary conditions on the interface are that the shear stress vanishes, that the normal stress is balanced by the force due to surface tension and that the interface is a stream surface. Hence

$$\left.\begin{aligned} p_{ns} &= 0, \\ p_{nn} &= T(1/\rho_1 + 1/\rho_2) \\ \psi &= 0 \end{aligned}\right\} \quad \text{when} \quad r = \epsilon R(z)\,(|z| \leqslant a). \tag{2.7}$$

Here T is the surface tension and ρ_1 and ρ_2 are the principal radii of curvature of the interface. Since the bubble is slender only because the surface tension is small, it can be anticipated that T and ϵ are related.

The components of the stress tensor are

$$\left.\begin{aligned} p_{rr} &= -p + 2\mu\, \partial v_r/\partial r, \\ p_{zz} &= -p + 2\mu\, \partial v_z/\partial z, \\ p_{rz} &= p_{zr} = \mu(\partial v_r/\partial z + \partial v_z/\partial r). \end{aligned}\right\} \tag{2.8}$$

Consequently, if θ is the angle between the normal to the bubble surface and the positive z axis, it follows that the shear stress on the interface is

$$p_{ns} = 2\mu \cos\theta \sin\theta \left(\frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r}\right) + \mu(\sin^2\theta - \cos^2\theta)\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right), \tag{2.9}$$

whereas the normal stress is

$$p_{nn} = -p + 2\mu\left(\cos^2\theta \frac{\partial v_z}{\partial z} + \sin^2\theta \frac{\partial v_r}{\partial r}\right) + 2\mu \cos\theta \sin\theta \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right). \tag{2.10}$$

The velocity derivatives are to be evaluated on the interface, of course.

The boundary conditions (2.7) will now be examined for the case when ϵ is small. Consider the last one, namely $\psi = 0$ on the interface. This is equivalent to

$$-\tfrac{1}{2}\epsilon^2 C z R^2(z) = \epsilon^2 R^2(z) \int_{-a}^{a} \frac{d\tilde{z}\, f(\tilde{z})}{[\epsilon^2 R^2(z) + (z-\tilde{z})^2]^{\frac{1}{2}}} - \int_{-a}^{a} d\tilde{z}\, \frac{g(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2 R^2(z) + (z-\tilde{z})^2]^{\frac{1}{2}}}. \tag{2.11}$$

The integrals that appear here must be expanded for small ϵ and we can do this provided that f and g are analytic, which is certainly true away from the ends. It can be anticipated that non-analyticity at the ends is unimportant provided that R vanishes in a roughly linear way, because as far as the integrals are concerned $\epsilon R(z)$ is the small parameter.

The method for asymptotically expanding integrals of this kind has been discussed by Handelsman & Keller (1967) in the context of potential theory and by Tillett (1970) in the context of Stokes flow. It consists of forming composite (uniformly valid) expansions of the integrand. Thus, anticipating the order of magnitude of f and g by writing $f = \epsilon^2 f_0$, $g = \epsilon^2 g_0$, equation (2.11) may be approximated by

$$\tfrac{1}{2}CzR^2(z) \sim \int_{-a}^{a} d\tilde{z}\, g_0(\tilde{z}) \frac{z-\tilde{z}}{|z-\tilde{z}|},$$

which, on differentiation, becomes

$$g_0(z) \sim \frac{C}{4}\frac{d}{dz}(zR^2). \tag{2.12}$$

Equations (2.9) and (2.10) for the interface stresses may be simplified when ϵ is small since $\sin\theta \sim 1$, $\cos\theta \sim -\epsilon R'(z)$. The shear-stress condition is then

$$0 \sim 3\epsilon CR' + 2\epsilon R' \left[\frac{2}{r}\frac{\partial^2\psi'}{\partial r\,\partial z} - \frac{1}{r^2}\frac{\partial\psi'}{\partial z}\right]_{r=\epsilon R} - \left[-\frac{1}{r}\frac{\partial^2\psi'}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi'}{\partial r}\right)\right]_{r=\epsilon R}. \tag{2.13}$$

Now,

$$\frac{1}{r}\frac{\partial^2\psi'}{\partial r\,\partial z}\bigg|_{r=\epsilon R} = \int_{-a}^{a} d\tilde{z}\frac{f(\tilde{z})\,(z-\tilde{z})\,[\epsilon^2R^2(z)-2(z-\tilde{z})^2]}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}} - \int_{-a}^{a} d\tilde{z}\frac{g(\tilde{z})\,[2(z-\tilde{z})^2-\epsilon^2R^2(z)]}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}}.$$

In order to see how these integrals behave, it is enough to observe that when ϵ is small the first integrand is proportional to something that looks like a differentiated delta function, whereas the second is like the delta function itself. A more precise approach is possible of course, as was previously indicated, but it is surely apparent that the first integral is at most $O(\epsilon)$ whereas the second could be $O(1)$, and

$$\frac{1}{r}\frac{\partial^2\psi'}{\partial r\,\partial z}\bigg|_{r=\epsilon R} \sim g(z)\int_{-a}^{a} d\tilde{z}\frac{\epsilon^2R^2(z)-2(z-\tilde{z})^2}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}}.$$

On changing the variable by putting $\tilde{z}-z = \epsilon Rt$, we have

$$\frac{1}{r}\frac{\partial^2\psi'}{\partial r\,\partial z}\bigg|_{r=\epsilon R} \sim \frac{g(z)}{\epsilon^2R^2}\int_{(-a-z)/\epsilon R}^{(a-z)/\epsilon R} dt\frac{(1-2t^2)}{(1+t^2)^{\frac{5}{2}}}$$
$$\sim \frac{g_0(z)}{R^2}\int_{-\infty}^{\infty} dt\frac{(1-2t^2)}{(1+t^2)^{\frac{5}{2}}}. \tag{2.14}$$

Actually, this last integral vanishes, so that it is not an $O(1)$ quantity. Notwithstanding, it is important to notice that replacing the integration limits by $\pm\infty$ is uniformly valid for small ϵ if R vanishes at the ends like $(a-|z|)$.

Also,

$$\frac{1}{r^2}\frac{\partial\psi'}{\partial z}\bigg|_{\epsilon R} = -\int_{-a}^{a} d\tilde{z}\frac{f(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{3}{2}}} - \int_{-a}^{a}\frac{d\tilde{z}\,g(\tilde{z})}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{3}{2}}}.$$

The first of these integrals is at most $O(\epsilon)$, so that

$$\frac{1}{r^2}\frac{\partial\psi'}{\partial z}\bigg|_{\epsilon R} \sim \frac{-g_0(z)}{R^2(z)}\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^{\frac{3}{2}}}. \tag{2.15}$$

Turning now to the equation

$$\frac{1}{r}\frac{\partial^2\psi'}{\partial z^2}\bigg|_{\epsilon R} = \int_{-a}^{a} d\tilde{z}\frac{f(\tilde{z})\,\epsilon R(z)\,[2(z-\tilde{z})^2-\epsilon^2R^2(z)]}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}} + 3\epsilon R(z)\int_{-a}^{a} d\tilde{z}\frac{g(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}},$$

we see that both of these integrals are $O(\epsilon)$ and so contribute to the shear-stress relation (2.13). Integrating by parts gives

$$\int_{-a}^{a} d\tilde{z}\, \frac{g(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2 R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}} = \frac{g(a)}{3[\epsilon^2 R^2+(z-a)^2]^{\frac{3}{2}}} - \frac{g(-a)}{3[\epsilon^2 R^2+(z+a)^2]^{\frac{3}{2}}}$$
$$-\frac{1}{3}\int_{-a}^{a} d\tilde{z}\, \frac{g'(\tilde{z})}{[\epsilon^2 R^2(z)+(z-\tilde{z})^2]^{\frac{3}{2}}}.$$

Since both g and f can be expected to vanish at the ends (and this can be confirmed later) it follows that

$$\frac{1}{r}\frac{\partial^2 \psi'}{\partial z^2}\bigg|_{\epsilon R} \sim \epsilon \frac{f_0}{R(z)}\int_{-\infty}^{\infty} dt\, \frac{2t^2-1}{(1+t^2)^{\frac{5}{2}}} - \epsilon \frac{g_0'}{R}\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\frac{3}{2}}}. \tag{2.16}$$

The final term in p_{ns} is

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi'}{\partial r}\right)\bigg|_{\epsilon R} = -\epsilon R \int_{-a}^{a} d\tilde{z}\, \frac{f(\tilde{z})\,[\epsilon^2 R^2+4(z-\tilde{z})^2]}{[\epsilon^2 R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}} - 3\epsilon R \int_{-a}^{a} d\tilde{z}\, \frac{g(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2 R^2(z)+(z-\tilde{z})^2]^{\frac{5}{2}}},$$

so that

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi'}{\partial r}\right)\bigg|_{\epsilon R} \sim -\epsilon \frac{f_0}{R}\int_{-\infty}^{\infty} dt\, \frac{(1+4t^2)}{(1+t^2)^{\frac{5}{2}}} + \epsilon \frac{g_0'}{R}\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\frac{3}{2}}}. \tag{2.17}$$

If all these expressions are substituted into the shear-stress condition (2.13), an equation relating f and g is obtained:

$$0 \sim 3CR^2R' + 4g_0 R' - 4g_0' R + 4f_0 R. \tag{2.18}$$

The expression for the normal stress, equation (2.10), may be treated in a similar fashion and, in fact, all the required integral approximations have already been made. Thus, on the interface

$$p_{nn} \sim -p - \mu C - 2\mu \frac{g_0}{R^2}\int_{-\infty}^{\infty} dt\, \frac{2-t^2}{(1+t^2)^{\frac{5}{2}}}. \tag{2.19}$$

An irrotational source generates no pressure disturbance, whereas a unit Stokeslet gives rise to a pressure $2\mu z/(z^2+r^2)^{\frac{3}{2}}$, so on the bubble surface

$$p = p_\infty + 2\mu \int_{-a}^{a} d\tilde{z}\, \frac{f(\tilde{z})\,(z-\tilde{z})}{[\epsilon^2 R^2(z)+(z-\tilde{z})^2]^{\frac{3}{2}}}, \tag{2.20}$$

which differs from p_∞ by a term of, at most, $O(\epsilon^2)$. Therefore the normal stress balance on the interface gives

$$-p_\infty - \mu C - 4\mu\, g_0/R^2 \sim T/\epsilon R, \tag{2.21}$$

where for pointed bubbles it has been recognized that the only important curvature is in planes of constant z. It is clear from (2.21) that the surface tension T is of $O(\epsilon)$, so that if we write $T = \epsilon \tilde{T}$ and eliminate g_0 between (2.12) and (2.21) the following equation for R is obtained:

$$zR' + (p_\infty/2\mu C + 1)\, R = -\tilde{T}/2\mu C. \tag{2.22}$$

In general, this equation will not have an analytic solution in the interval $[-a, a]$, but if p_∞ is chosen so that

$$p_\infty/2\mu C + 1 = -n, \tag{2.23}$$

where n is an even positive integer, the solution for R is

$$R = (\tilde{T}/2\mu Cn)[1-(z/a)^n]. \tag{2.24}$$

It is no surprise that p_∞ is determined as part of the solution, since the pressure inside the bubble was arbitrarily chosen to be zero. However, there is no obvious criterion for determining the parameter n. It is tempting to assume that n takes the smallest possible value, namely 2. The bubble is then the thickest possible for a given value of $\tilde{T}/\mu Ca$ and the pressure difference between the far field and the bubble interior is a minimum. Furthermore, the bubbles photographed by both Taylor and Rumscheidt & Mason have shapes consistent with z^2 behaviour, rather than the blunter shapes associated with larger choices of n. However, there is no known logically based principle justifying the choice $n = 2$, although since the bubbles for different n are more slender they might be expected to be unstable. Certainly the extension of Helmholtz's minimum dissipation principle to flows with free surfaces derived by Keller, Rubenfeld & Molyneux (1967) cannot be applied to the present problem, since their theory assumes that the free surface is fixed during the variation.

The result (2.24) when $n = 2$ is actually given by Taylor (1964) as a limiting case of an analysis of drops of arbitrary viscosity. Unfortunately this analysis is not given and there is no clear statement of the assumptions that underly it. The only clues are an analysis of flow over a slender cone together with a statement that the drop analysis follows similar lines. Consequently it is not possible to decide how the contribution of the pressure to the normal stress was handled, and this appears to make an essential contribution to the non-uniqueness revealed in the present paper. It might be thought that Taylor's inclusion of drop viscosity resolves the non-uniqueness, but this is not true. The extension of the present analysis to include interior viscosity has been made, including the unsteady description of a bursting drop, and the results of this analysis are also non-unique. They are not included here since they form part of a separate discussion of bursting drops† (Buckmaster 1972), whereas here we are solely concerned with the question of points in drop interfaces.

Once R has been determined, f_0 and g_0 can be found:

$$\left.\begin{aligned} f_0(z) &= -C\frac{n(n-1)}{2a^2}\left(\frac{\tilde{T}}{2\mu Cn}\right)^2\left(\frac{z}{a}\right)^{n-2}\left[1-\left(\frac{z}{a}\right)^n\right], \\ g_0(z) &= \frac{C}{4}\left(\frac{\tilde{T}}{2\mu Cn}\right)^2\left[1-\left(\frac{z}{a}\right)^n\right]\left[1-\left(\frac{z}{a}\right)^n-2n\left(\frac{z}{a}\right)^n\right], \end{aligned}\right\} \tag{2.25}$$

and these vanish at the ends, as was previously assumed.

The bubble described by (2.24) has conical ends which, although consistant with the experimental evidence, contradicts the result of the appendix, where

† The ability of our model to predict bursting when drop viscosity is included, in agreement with experiment, provides strong evidence that the model is realistic.

it is shown that no local solution exists in the neighbourhood of a conical interface. In order to clarify this apparent paradox, the second-order solution is considered in the next section.

3. Second-order solution

The surface tension should be regarded as a given $O(\epsilon)$ quantity. The solution to the bubble problem can then be written in the form

$$\left.\begin{aligned} R(z) &\sim R_0(z)+\epsilon^2 R_1(z)+\dots,\\ f(z) &\sim \epsilon^2 f_0(z)+\epsilon^4 f_1(z)+\dots,\\ g(z) &\sim \epsilon^2 g_0(z)+\epsilon^4 g_1(z)+\dots,\\ p_\infty &\sim (p_\infty)_0+\epsilon^2(p_\infty)_1+\dots.\end{aligned}\right\} \tag{3.1}$$

Dependence on ϵ other than that explicitly shown is permitted provided that it is much weaker than algebraic. It turns out, in fact, that logarithmic terms appear. The procedure for determining the quantities in (3.1) follows that of § 2 except that the various integrals must be asymptotically evaluated to a higher degree of accuracy, and this involves forming composite expansions for the integrands as described by Tillett (1970). The details are straightforward but lengthy and it is felt that presenting them in full is not justified. Instead, the final results needed to develop an equation for $R_1(z)$ will be presented.

The condition that the bubble boundary is a stream surface leads to the result

$$CzR_0(z)\,R_1(z)=\int_{-a}^{a}d\tilde z\,g_1(\tilde z)\frac{z-\tilde z}{|z-\tilde z|}-\mathscr{F}(z), \tag{3.2a}$$

where

$$\begin{aligned}\mathscr{F}(z)\equiv R_0^2\left[\int_{-a}^{a}d\tilde z\,\frac{f_0(\tilde z)-f_0(z)}{|z-\tilde z|}+f_0(z)\ln\left(\frac{4(a^2-z^2)}{\epsilon^2R_0^2}\right)\right]\\ -\tfrac{1}{2}g_0R_0^2\left(\frac{1}{z+a}+\frac{1}{z-a}\right)+\tfrac{1}{2}g_0'R_0^2\left[1-\ln\left(\frac{4(a^2-z^2)}{\epsilon^2R_0^2}\right)\right]\\ -\tfrac{1}{2}R_0^2\int_{-a}^{a}\frac{d\tilde z}{(z-\tilde z)\,|z-\tilde z|}\left(-g_0(\tilde z)+g_0(z)+(\tilde z-z)\,g_0'(z)\right).\end{aligned} \tag{3.2b}$$

Note that $\mathscr{F}(z)$ only depends on the leading solutions R_0, f_0 and g_0. An explicit representation as a function of z is possible but will not be needed for our purposes.

In addition, we have the following results:

$$p\sim(p_\infty)_0+\epsilon^2(p_\infty)_1+\epsilon^2\alpha(z), \tag{3.3a}$$

$$\alpha(z)\equiv-2\mu\left[\int_{-a}^{a}d\tilde z\,\frac{f_0'(\tilde z)-f_0'(z)}{|z-\tilde z|}+f_0'(z)\ln\left(\frac{4(a^2-z^2)}{\epsilon^2R_0^2}\right)\right], \tag{3.3b}$$

$$\partial v_z/\partial z\sim C, \tag{3.4}$$

$$\partial v_r/\partial z\sim 2\epsilon\,g_0'/R_0, \tag{3.5}$$

$$\partial v_z/\partial r\sim-4\epsilon f_0/R_0+2\epsilon\,g_0'/R_0, \tag{3.6}$$

$$\frac{\partial v_r}{\partial r}=-\frac{C}{2}-\frac{1}{r}\frac{\partial^2\psi'}{\partial r\,\partial z}+\frac{1}{r^2}\frac{\partial\psi'}{\partial z}, \tag{3.7a}$$

where

$$\frac{1}{r^2}\frac{\partial\psi'}{\partial z} \sim -2\frac{g_0}{R_0^2} - 2\epsilon^2\frac{g_1}{R_0^2} + 4\epsilon^2\frac{g_0 R_1}{R_0^3} + \epsilon^2\beta(z) \tag{3.7b}$$

and

$$\begin{aligned}\beta(z) \equiv & \left[\int_{-a}^{a} d\tilde{z}\,\frac{f_0'(\tilde{z})-f_0'(z)}{|z-\tilde{z}|} + f_0'(z)\ln\left(\frac{4(a^2-z^2)}{\epsilon^2 R_0^2}\right)\right] \\ & -\int_{-a}^{a} d\tilde{z}\,\frac{[g_0(\tilde{z})-g_0(z)-(\tilde{z}-z)\,g_0'(z)-\tfrac{1}{2}(\tilde{z}-z)^2 g_0''(z)]}{|z-\tilde{z}|^2} \\ & +\tfrac{1}{2}g_0\left(\frac{1}{(a-z)^2}+\frac{1}{(a+z)^2}\right) + g_0'(z)\left(\frac{1}{a-z}-\frac{1}{a+z}\right) + \tfrac{1}{2}g_0''\left[2-\ln\left(\frac{4(a^2-z^2)}{\epsilon^2 R_0^2}\right)\right].\end{aligned} \tag{3.7c}$$

Also,

$$\frac{1}{r}\frac{\partial^2\psi'}{\partial r\,\partial z} \sim \epsilon^2\gamma(z), \tag{3.8a}$$

where

$$\begin{aligned}\gamma(z) = & \,2\left[\int_{-a}^{a} d\tilde{z}\,\frac{f_0'(\tilde{z})-f_0'(z)}{|z-\tilde{z}|} - f_0'(z) + f_0'(z)\ln\left(\frac{4(a^2-z^2)}{\epsilon^2 R_0^2}\right)\right] \\ & -2\int_{-a}^{a} d\tilde{z}\,\frac{g_0(\tilde{z})-g_0(z)-(\tilde{z}-z)\,g_0'(z)-\tfrac{1}{2}(\tilde{z}-z)^2 g_0''(z)}{|z-\tilde{z}|^3} \\ & +g_0\left(\frac{1}{(a-z)^2}+\frac{1}{(a+z)^2}\right) + 2g_0'\left(\frac{1}{a-z}-\frac{1}{a+z}\right) + g_0''\left[3-\ln\left(\frac{4(a^2-z^2)}{\epsilon^2 R_0^2}\right)\right].\end{aligned} \tag{3.8b}$$

Now the normal stress balance on the interface, including terms of order ϵ^2, gives

$$\begin{aligned}-p + 2\mu\epsilon^2 R_0'^2\frac{\partial v_z}{\partial z} + 2\mu(1-\epsilon^2 R_0'^2)\frac{\partial v_r}{\partial r} - 2\mu\epsilon R_0'\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) \\ \sim \frac{\tilde{T}}{R_0} - \epsilon^2\tilde{T}\left(R_0''+\frac{R_1}{R_0^2}+\frac{R_0'^2}{2R_0}\right).\end{aligned} \tag{3.9}$$

Note that the curvature in the azimuthal plane contributes to $O(\epsilon^2)$.

Collecting $O(\epsilon^2)$ terms in (3.9), we find that

$$-(p_\infty)_1 R_0^2 + 2\mu(-2g_1 + 4g_0 R_1/R_0) = -\tilde{T}R_1 + \mathscr{G}(z)\,R_0^2, \tag{3.10a}$$

where

$$\begin{aligned}\mathscr{G}(z) = & -\tilde{T}R_0'' - \tilde{T}\,R_0'^2/2R_0 + \alpha(z) - \mu R_0'^2(3C + 4\,g_0/R_0^2) \\ & -2\mu[\beta(z)-\gamma(z)] + 2\mu R_0'(-4f_0/R_0 + 4g_0'/R_0).\end{aligned} \tag{3.10b}$$

If g_1 is eliminated between (3.2a) and (3.10b) an equation for R_1 is obtained:

$$\frac{dR_1}{dz} - n\frac{R_1}{z} = -n\frac{H(z)}{\tilde{T}z[1-(z/a)^n]} \equiv Q(z), \tag{3.11a}$$

where

$$H(z) \equiv R_0^2\mathscr{G}(z) + (p_\infty)_1 R_0^2 + 2\mu\, d\mathscr{F}/dz. \tag{3.11b}$$

$H(z)$ is an even function of z and vanishes at the ends like $(a-|z|)\ln(a-|z|)$ (contributed by $d\mathscr{F}/dz$), so that $Q(z)$ is integrable at the ends. It follows that for $-a \leqslant z < 0$

$$R_1(z) = z^n\int_{-a}^{z} ds\,\frac{Q(s)}{s^n}. \tag{3.12}$$

The solution for R_1 will be analytic at the origin only if the expansion of $Q(s)/s^n$ for small s does not contain a $(1/s)$ term. Since $(p_\infty)_1$ contributes to the coefficient of this term, a unique choice of $(p_\infty)_1$ will ensure that R_1 has an acceptable behaviour. Thus no additional non-uniqueness is introduced at the second order. Once analyticity has been enforced, the evenness of H ensures that all the odd derivatives of R_1 vanish as $z \to 0$, so that, by reflexion, R_1 is properly defined over the whole interval $[-a, a]$.

Two things should be noticed about this solution. First, it tells us nothing about n and second, since $Q(z) \sim \ln(a-|z|)$ at the ends, $R_1(z)$ behaves like $(a-|z|)\ln(a-|z|)$. Later terms in the expansion will contain higher powers of $\ln(a-|z|)$, so that since $R_0(z)$ behaves like $(a-|z|)$ this result suggests that in an $O(e^{-1/\epsilon^2})$ neighbourhood of the ends the present solution is not valid. This interpretation resolves the apparent contradiction between §2, which predicts conical ends, and the appendix, where conical ends are discredited. In addition it is consistent with the experimental evidence since, in the thickest bubbles with pointed ends shown, $O(e^{-1/\epsilon^2})$ regions would be completely indistinguishable.

The question of what happens in these exponentially small regions remains. Several possibilities were mentioned in §1 but it bears repeating that, in the absence of either a thorough analysis or experimental evidence, any discussion is necessarily speculative. However, if a steady solution in these tiny regions is possible, a local solution near the ends must exist. It is suggestive, then, that in the next section the first term in a local solution near a cusped end is derived. It does not follow that such cusps are a reality, but they are a possibility.

4. Local solution near a cusp

Consider an interface between a vacuum and a viscous fluid with shape (see figure 2)

$$z = Kr^{\alpha} \quad (0 < \alpha < 1) \tag{4.1}$$

near the origin. We want to see whether we can derive a local solution of the Stokes equations satisfying all the conditions on the bubble surface, in addition

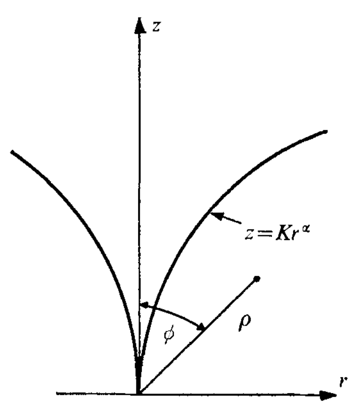

FIGURE 2. Cusped end.

to being analytic outside the bubble. It is not obvious that such a solution exists, as the appendix shows for the special case $\alpha = 1$. We shall not consider the case of arbitrary α (between the prescribed limits) since a solution has not been found for the general case. Matching difficulties involving logarithmic terms occur and it does not seem possible to decide whether this is because a solution does not exist in the general case or because a local solution of the appropriate form was not attempted. It does not seem probable that a local solution exists for all α. However, when $\alpha = 1/n$, where n is a positive integer, a local solution can be started and it is this solution that we shall describe.

The normal stress on the interface $\sim 1/r$ for small r which suggests that close to the interface the pressure has the form

$$p \sim (1/r)P_i(\eta), \quad \eta \equiv z/r^\alpha. \tag{4.2}$$

Now p satisfies Laplace's equation, which in terms of η and r is

$$0 = \frac{\partial^2 p}{\partial r^2} + \left(\frac{\alpha^2\eta^2}{r^2} + \frac{1}{r^{2\alpha}}\right)\frac{\partial^2 p}{\partial \eta^2} - 2\frac{\alpha\eta}{r}\frac{\partial^2 p}{\partial r\,\partial \eta} + \frac{\alpha^2\eta}{r^2}\frac{\partial p}{\partial \eta} + \frac{1}{r}\frac{\partial p}{\partial r}.$$

It follows that $P_i(\eta)$ satisfies the equation

$$\alpha^2\eta^2 P_i'' + (2\alpha + \alpha^2)\,\eta P_i' + P_i = 0, \tag{4.3}$$

which has general solution

$$P_i = A\eta^{-1/\alpha} + B\eta^{-1/\alpha}\ln\eta. \tag{4.4}$$

This is only acceptable when $\eta > 0$, so that different variables are needed once the angle ϕ is large enough, (ρ, ϕ) being spherical polar co-ordinates. In fact it seems reasonable to suppose that for fixed ρ the flow is unaware of the interface curvature except when ϕ is small. Consequently the spherical polar co-ordinates (ρ, ϕ) are appropriate in most of the region. The equation for p is then

$$\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial p}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial p}{\partial \phi}\right) = 0. \tag{4.5}$$

Now on the interface $p \sim 1/r \sim z^{-1/\alpha}$, and for small ϕ, z is indistinguishable from ρ, so the appropriate form of the outer solution is

$$p \sim \rho^{-1/\alpha}P_0(\phi). \tag{4.6}$$

Substitution into (4.5) then yields the following equation for P_0:

$$\sin\phi P_0'' + \cos\phi P_0' + (1/\alpha)(1/\alpha - 1)\sin\phi P_0 = 0, \tag{4.7}$$

with general solution

$$P_0 = EP_{1/\alpha - 1}(\cos\phi) + DQ_{1/\alpha - 1}(\cos\phi), \tag{4.8}$$

where P_ν and Q_ν are Legendre functions. If $1/\alpha$ is an integer, P_ν is analytic at $\phi = \pi$ whereas Q_ν has logarithmic behaviour, so we must choose $D = 0$. Matching with the inner solution then yields the following description for the pressure:

$$\left.\begin{aligned} p_i &\sim (c/r)\,(\eta/K)^{-1/\alpha}, \\ p_0 &\sim cK^{1/\alpha}\rho^{-1/\alpha}P_{1/\alpha - 1}(\cos\phi), \end{aligned}\right\} \tag{4.9}$$

where c is the value of pr on the bubble surface.

Let us now calculate the velocity field consistent with this pressure. In the inner region the velocity components must have the form

$$v_r = f(\eta), \quad v_z = r^{\alpha-1}g(\eta). \tag{4.10}$$

It follows that the leading terms in the expansions of the velocity for small r satisfy the equations

$$\Delta v_r - v_r/r^2 = 0, \quad \Delta v_z = 0,$$

together with the continuity equation

$$0 = \frac{\partial v_r}{\partial r} - \frac{\alpha\eta}{r}\frac{\partial v_r}{\partial \eta} + \frac{v_r}{r} + \frac{1}{r^\alpha}\frac{\partial v_z}{\partial \eta},$$

so that

$$\left.\begin{aligned} \alpha^2\eta^2 g'' + (2\alpha-\alpha^2)\,\eta g' + (\alpha-1)^2 g &= 0,\\ \alpha^2\eta^2 f'' + \alpha^2\eta f' - f &= 0.\end{aligned}\right\} \tag{4.11}$$

The general solution satisfying continuity is then

$$f(\eta) = S\eta^{-1/\alpha} + R\eta^{1/\alpha}, \quad g(\eta) = \frac{2S}{(1/\alpha-1)}\eta^{1-1/\alpha}, \tag{4.12}$$

where S and R are constants. Additional restraints are imposed on this inner solution by the boundary conditions on the interface. Thus the tangency condition

$$[v_z/v_r]_{\eta=K} = \alpha K r^{\alpha-1}$$

is satisfied only if

$$R = \left(\frac{1+\alpha}{1-\alpha}\right)K^{-2/\alpha}S. \tag{4.13}$$

Also, the normal stress at the interface is

$$p_{nn} = \frac{1}{r}\left\{-P_i - 2\mu\alpha\eta f' - \frac{2\mu}{\alpha K}[(\alpha-1)\,g - \alpha\eta g']\right\}_{\eta=K} = \frac{T}{r},$$

where T is the surface tension, whence

$$T = -c + 2\mu S K^{-1/\alpha} - 2\mu S\left(\frac{1+\alpha}{1-\alpha}\right)K^{-1/\alpha}. \tag{4.14}$$

Finally the shear stress at the interface is

$$p_{ns} = -\frac{\mu}{r^{2-\alpha}}[(\alpha-1)\,g - \alpha\eta g']_{\eta=K} = 0;$$

this is identically satisfied by the function $g(\eta)$. Thus the complete inner solution, satisfying all conditions on the interface to leading order, is

$$\left.\begin{aligned} p &\sim \left\{-T + 2\mu S K^{-1/\alpha} - 2\mu S\left(\frac{1+\alpha}{1-\alpha}\right)K^{-1/\alpha}\right\}\frac{K^{1/\alpha}}{r}\eta^{-1/\alpha},\\ v_r &\sim S\left\{\eta^{-1/\alpha} + \left(\frac{1+\alpha}{1-\alpha}\right)K^{-2/\alpha}\eta^{1/\alpha}\right\},\\ v_z &\sim \frac{2S\alpha}{1-\alpha}r^{\alpha-1}\eta^{1-1/\alpha}.\end{aligned}\right\} \tag{4.15}$$

This is to be matched with the outer solution which can be described in terms of the stream function:

$$v_\rho = \frac{1}{\rho^2 \sin\phi}\frac{\partial\psi}{\partial\phi}, \quad v_\phi = -\frac{1}{\rho\sin\phi}\frac{\partial\psi}{\partial\rho},$$

where

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\sin\phi}{\rho^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\right)\right]^2\psi = 0.$$

The most general solution analytic at $\phi = \pi$ is

$$\psi = \rho^{3-1/\alpha}\{L\sin^2\phi P'_{1/\alpha-3}(\cos\phi)+M\sin^2\phi P'_{1/\alpha-1}(\cos\phi)\}. \tag{4.16}$$

Matching of both v_r and v_z with the inner solution is accomplished if

$$\frac{S\alpha}{1-\alpha} = LP'_{1/\alpha-3}(1)+MP'_{1/\alpha-1}(1), \tag{4.17}$$

and in addition, since the pressure and velocities are related by the momentum equation,

$$cK^{1/\alpha} = \mu M(1/\alpha-1)(4/\alpha-6). \tag{4.18}$$

These conditions determine L and M in terms of α, K and S, so the complete local solution, to first order, depends on these three undetermined parameters. We have suggested that $1/\alpha$ is a positive integer, so that this aspect of the non-uniqueness is reminiscent of the non-uniqueness in § 2 (see equation (2.24)). Also, K is a measure of the thickness ratio of the bubble and since this is determined by the strain in the far field, which a local analysis cannot be aware of, it is no surprise that this appears as an undefined parameter. It is less clear why there is a third unknown parameter, S. However, an examination of the solution for the pressure over the major portion of the bubble given by (3.3) shows that this is not algebraically singular as the ends are approached; the singular part of the normal stress is derived entirely from the velocity field. If this result is also true at the ends then S is determined by requiring the first of (4.15) to vanish.

It should not be thought that the solution described here is appropriate for all integer values of $1/\alpha$. The radial velocity in the outer flow field is

$$v_\rho = \rho^{1-1/\alpha}\{2L\cos\phi P'_{1/\alpha-3}(\cos\phi)-L\sin^2\phi P''_{1/\alpha-3}(\cos\phi) + 2M\cos\phi P'_{1/\alpha-1}(\cos\phi)-M\sin^2\phi P''_{1/\alpha-1}(\cos\phi)\}.$$

Therefore if $1/\alpha$ is odd, the value of v_ρ for some ϕ in $(0, \frac{1}{2}\pi)$ is the same as that at $\phi+\frac{1}{2}\pi$ and so does not correspond to flow off the end of the bubble.† Consequently only solutions for *even* values of $1/\alpha$ are acceptable, and the analogy with the earlier sections is even more striking. Remember, however, that it was never shown that solutions for non-integral values of $1/\alpha$ do not exist.

The solution obtained here is, of course, singular at the tip. This is not physically acceptable but there seems to be no reason why the difficulty could not be resolved on a molecular scale. That is, just as a shock wave is only a discontinuity on a continuum scale, so presumably is the present flow only singular on such a scale. Thus the singularity is not an adequate reason for dismissing the present cusped solutions in favour of a rounded solution. The regions involved are apparently too small for this question to be resolved by experiment.

† I am grateful to Professor Lu Ting for pointing this out.

5. Concluding remarks

In view of our inability to model faithfully the experimental situation, it seems worth while to list the essential qualitative features observed by both Taylor and Rumscheidt & Mason and compare these with the predictions of the axisymmetric model. They found that relatively inviscid drops, if strained sufficiently, exhibited pointed ends and that these drops survived up to the maximum rates of strain attainable in the apparatus. On the other hand, viscous drops, whilst also displaying pointed ends, burst if the straining was too great. The present work deals only with inviscid drops, and in § 2 we saw that pointed bubbles are predicted, in the large. Furthermore, solutions exist for all values of $\tilde{T}/\mu Ca$. These results agree with the experimental observations of relatively inviscid drops. Furthermore, the author has discussed viscous drops (Buckmaster 1972) using the axisymmetric model, and has shown that pointed solutions exist (in the large) and burst if the straining is too great, in agreement with experiment. There is good reason to believe, therefore, that the axisymmetric model retains the essential physics.

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Appendix. Failure of a local conical solution

It is to be expected that the solution for ψ can locally be represented by a sum of terms separable in ρ and ϕ. Of particular interest is the term which provides the normal stress necessary to balance that associated with the surface tension at the interface. This term is

$$\psi = \rho^2 T(\phi). \tag{A 1}$$

The slow-flow equations hence yield the following general solution analytic exterior to the bubble:

$$\psi = \rho^2[a\cos\phi + b + c\sin^2\phi], \tag{A 2}$$

where a, b and c are constants. Since ψ must be a constant when $\phi = \pi$, it follows that

$$a = b, \tag{A 3}$$

so that there are only two undetermined constants left to satisfy the three conditions on the interface.

The pressure associated with the velocity field is

$$p = 2\mu a/\rho, \tag{A 4}$$

from which it follows that the stress

$$p_{\phi\phi} = -p + 2\mu\left(\frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\rho}{\rho}\right) = \frac{4\mu}{\rho}\frac{\cos\phi}{\sin^2\phi}(a\cos\phi + b). \tag{A 5}$$

Also,

$$p_{\rho\phi} = \mu\left(\rho\frac{\partial}{\partial\rho}\left(\frac{v_\phi}{\rho}\right) + \frac{1}{\rho}\frac{\partial v_\rho}{\partial\phi}\right) = \frac{2\mu}{\rho\sin\phi}(a\cos\phi + b). \tag{A 6}$$

In view of (A 3) there is no way we can ensure that the shear stress vanishes at the interface, and even if we could the normal stress would then vanish also, which is not acceptable. We conclude that a pointed bubble is not conical. Taylor (1964), in his approximate analysis of flow over a slender cone, nowhere considered the shear balance, which is an essential part of the present argument. Consequently his work does not reveal the inability of the point to remain truly conical. A statement is made that more careful considerations suggest the inadequacy of a conical point, but no details are given.

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